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Smooth Extension of Functions Known at Discrete Points—Application to Magnetic Signature Measurements

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Magnetic signature measurements of tanks at Aberdeen, MD, are taken that effectively measure the vertical field in a plane below the tank at points spaced 1 ft apart in a 20 × 40 ft grid. By use of a discrete Fourier series expansion, the measurements A(pq) are extended to arbitrary points within the area by R(uv) where R(pq) = A(pq) for integer p,q. This extension allows one to compare sets of measurements that are slightly shifted with respect to one

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CONTENTS

1. INTRODUCTION

At the Harry Diamond Laboratories (HDL) magnetic signature facility at Aberdeen, MD, magnetic signatures of tanks are determined by measuring the field in a plane under the tank at points spaced 1 ft apart in a 20 x 40 ft square grid (J.I. Cooperman and T. H. Hopp, unpublished report). Forty-one readings are taken (one for each foot) as the tank advances over a linear array of 21 magnetometers. The problem treated here is how best to correlate different sets of measurements to eliminate statistical uncertainties in the measurement. In a given run, the tank may take a path that is slightly shifted with respect to its previous run, and the process of recording readings on the magnetometers for every foot of travel of the tank across the linear array may have begun at a slightly different time. One cannot, therefore, simply compare the output of corresponding magnetometers and treat the problem by the usual statistical methods. There are good reasons, owing to the small shifts described above, that corresponding magnetometers *should not* give the same reading during different runs.

A second problem related to these measurements is sorting out the tank's distortion of the earth's magnetic field from the larger residual field of the tank. The latter is easily measured by bucking out the earth's field with large Helmholtz coils existing at the site, but the former must be inferred by comparing these measurements with similar measurements when the earth's field is not bucked out. Shifts similar to those described above must again be considered when one compares these measurements.

Both problems are treated here in a unified manner by considering the question, "What can be said about a function of two variables if it is sampled only at a finite number of points in a square grid covering a given area?" This problem is solved by expanding the matrix of measurements in a discrete Fourier series with the largest number of low-frequency components that can uniquely be determined from the measurements. This expansion provides a natural and uncontrived extension of the function continuously and smoothly to any point within the area. By comparing the functions generated in this manner from two separate runs, one may then continuously shift the functions with respect to each other to obtain a best fit. In this way, an analytic determination of the previously unknown shifts between the two runs is obtained, and systematic errors are eliminated that result when such shifts are ignored. Assuming the shifts to be small compared with the 1-ft spacings between data points, a closed expression for the shifts is obtained. Larger shifts may be determined by using well-known computer methods for minimizing the explicit expression derived for the square deviation between the functions.

The second problem may be treated similarly by comparing the two runs with and without the earth's field bucked out, except there is a question as to how the comparison should be made. One can only assume that the correlation between the tank's residual field and its distortion of the earth's field is small (if not zero) compared with its self correlation, so also a best fit between the shifted functions representing the two runs should be required in this case.* The remaining field after the best fit is subtracted is then the distorted earth's field, provided it may be assumed also that the magnetic behavior of the tank is linear for field strengths considered in these experiments. Although the formalism developed in this report may be useful in considering possible nonlinear effects, this problem is not treated here.

In formulating the mathematics to handle the problem discussed above, the larger question has been answered of determining a function of two variables by sampling at a discrete number of points in a regular grid. The methods in this report therefore more generally apply.

**In a private communication, T. H. Hopp of HDL suggests that there is reason to believe the distorted earth's field has a component that correlates in phase with the tank's residual field. This correlation provides even stronger justification for requiring a best-fit match.*

2. DISCRETE FOURIER EXPANSION

Let a sequence of readings on N detectors taken at M equal time or space intervals be stored in an NXM array A, where A(pq) is the q^{th} reading of the p^{th} detector. For convenience, start the numbering at zero so that

$$0 \leqslant p \leqslant N - 1 , \tag{1a}$$

$$0 \leqslant q \leqslant M - 1 . \tag{1b}$$

In this problem, A(uv) is a function of two variables, u and v, known only at integer values of its arguments over the finite ranges given above. To approximate the function for arbitrary uv, A(pq) is expanded in finite, discrete Fourier series and then generalized.

Define the Fourier components in the expansion of A by

$$A(pq) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} e^{i\frac{2\pi}{N} pn} e^{i\frac{2\pi}{M} qm} a_{nm} \tag{2}$$

where, from

$$\sum_{p=0}^{N-1} e^{i\frac{2\pi}{N} p(n-n')} = N\delta_{nn'} , \tag{3}$$

expression (2) may be inverted to give

$$a_{nm} = (NM)^{-1} \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} e^{-i\frac{2\pi}{N} pn} e^{-i\frac{2\pi}{M} qm} A(pq) . \tag{4}$$

To be consistent with this expansion, it is assumed that for p and q outside the ranges given by expressions (1a) and (1b), A is defined by

$$A(p+N, q) = A(pq) = A(p, q+M) . \tag{5}$$

Henceforth, it is assumed also that the implied summations in Σ_p and Σ_q, respectively, extend over the ranges given by expressions (1a) and (1b) and similarly in Σ_n and Σ_m, unless otherwise noted. Now define the extension of A to arbitrary uv by substituting p = u and q = v in expression (2) and using the definition of a_{nm} given by expression (4). Thus,

$$A(uv) = (NM)^{-1} \sum_{p'q'} A(p'q') \sum_{nm} e^{i\frac{2\pi}{N}(u-p')n} e^{i\frac{2\pi}{M}(v-q')m}$$

$$= e^{i\pi(u+v)} \sum_{p'q'} A(p'q') C_N (p'-u) C_M (q'-v) \tag{6}$$

where

$$C_N(p, z) \equiv N^{-1} \sin \pi z \frac{e^{i\frac{2\pi}{N}(p+z)}}{\sin \frac{2\pi}{N}(p+z)} \tag{7a}$$

$$= N^{-1} \sin \pi z \left[\cot \frac{2\pi}{N}(p+z) + i\right] \tag{7b}$$

The C_N functions are a convenient form for evaluating the summations needed later. In appendix A, sums over products of cotangents are discussed, and results are given from which useful properties of the C_N can be derived. These properties of the C_N are given in appendix B.

From expression (6), it is clear that A(uv) is not necessarily real for noninteger u and v. Therefore, define R(uv), the extention of A(pq) to noninteger arguments, as the real part of expression (6), to give

$$R(uv) = (NM)^{-1} \sin \pi u \sin \pi v \sum_{p'q'} A(p'q') \frac{\cos \pi \left[\left(1-\frac{1}{N}\right) u + \frac{p'}{N} + \left(1-\frac{1}{M}\right) v + \frac{q'}{M}\right]}{\sin \frac{\pi}{N}(p'-u) \sin \frac{\pi}{M}(q'-v)} \tag{8}$$

where this expression has been obtained by substituting expression (7a) for the C_N. Letting $u = p + x/2$, $v = q + y/2$, the following expression results:

$$R(p + x/2, q + y/2) = (NM)^{-1} \sin \frac{\pi x}{2} \sin \frac{\pi y}{2} \sum_{p'q'} A(p'q') \frac{\cos \pi \left[\frac{x}{2} + \frac{p'-p-x/2}{N} + \frac{y}{2} + \frac{q'-q-y/2}{M}\right]}{\sin \frac{\pi}{N}(p'-p-x/2) \sin \frac{\pi}{M}(q'-q-y/2)}. \tag{9}$$

This form is convenient for examining the relationship of R to A for small deviations x,y. As x and y approach zero, $R(pq) \rightarrow A(pq)$.

Similarly, if S is the analogous extension of a second set of measurements stored in an array B(pq), then the two extensions R and S may be compared by shifting S in the opposite direction and minimizing their square deviation. In this way, the two sets of measurements A and B are on equal footing. Thus, define

$$Q(x,y) = (NM)^{-1} \sum_{pq} \left[R(p + x/2, q + y/2) - S(p - x/2, q - y/2)\right]^2 \tag{10}$$

where S is given by letting x and y go to -x and -y and substituting B for A in expression (9). Carrying out the algebra according to the cotangent sums in appendix A (or more simply, the C_N relationships given in appendix B), one gets

$$\begin{aligned} Q(x,y) = {} & [1 + \cos \pi(x + y)]\, Q_1 + [\cos \pi y - \cos \pi(x + y)]\, Q_2 \\ & + [\cos \pi x - \cos \pi(x + y)]\, Q_3 + [1 + \cos \pi(x + y) - \cos \pi x - \cos \pi y]\, Q_4 \\ & - [1 + \cos \pi(x + y)] \sin \pi x \sin \pi y \, Q_5(x,y) \\ & - [1 - \cos \pi x + \sin \pi x \sin \pi(x + y)] \sin \pi y \, Q_6(y) \\ & - [1 - \cos \pi y + \sin \pi y \sin \pi(x + y)] \sin \pi x \, Q_7(x) \\ & - [(1 - \cos \pi x)(1 - \cos \pi y) - \sin \pi x \sin \pi y \cos \pi(x + y)]\, Q_8 \end{aligned} \tag{11}$$

where

$$Q_1 = \frac{1}{2}(NM)^{-1} \sum_{pq} \left[A^2(pq) + B^2(pq)\right], \tag{12a}$$

$$Q_2 = \frac{1}{2} N^{-2} M^{-1} \sum_{pp'q} \left[A(pq)A(p'q) + B(pq)B(p'q)\right], \tag{12b}$$

$$Q_3 = \frac{1}{2} N^{-1} M^{-2} \sum_{pqq'} \left[A(pq)A(pq') + B(pq)B(pq')\right], \tag{12c}$$

$$Q_4 = \frac{1}{2}\left(\bar{A}^2 + \bar{B}^2\right), \tag{12d}$$

$$Q_5(x,y) = (NM)^{-1} \sum_{pq} \cot\frac{\pi}{N}(p-x)\cot\frac{\pi}{M}(q-y)\,P(pq), \tag{12e}$$

$$Q_6(y) = (NM)^{-1} \sum_{q} \cot\frac{\pi}{M}(q-y) \sum_{p} P(pq), \tag{12f}$$

$$Q_7(x) = (NM)^{-1} \sum_{p} \cot\frac{\pi}{N}(p-x) \sum_{q} P(pq), \tag{12g}$$

$$Q_8 = \bar{A}\bar{B}, \tag{12h}$$

and where

$$\bar{A} = (NM)^{-1} \sum_{pq} A(pq), \tag{13}$$

and the correlation matrix P is defined by

$$P(pq) = (NM)^{-1} \sum_{p'q'} A(p'q')\,B(p'+p, q'+q). \tag{14}$$

In principle, the problem is now straightforward. Expression (11) must be minimized with respect to x and y—say this occurs at $x = x_o$, $y = y_o$; then the most probable function F (given the two sets of measurements) is

$$F(uv) = \frac{1}{2}\left[R(u + x_o/2, v + y_o/2) + S(u - x_o/2, v - y_o/2)\right] \tag{15}$$

with an rms deviation σ given by

$$\sigma = \left[Q(x_o, y_o)\right]^{1/2}. \tag{16}$$

Minimizing Q(xy) may be a formidable task even for a computer if N and M are of appreciable size; so the problem, although well defined, is not necessarily satisfactorily solved at this stage. In addition, there may be several sets of measurements, in which case the procedure outlined above must be performed several times and the results somehow correlated. In the following section, the case is treated where x_o and y_o may be assumed small in comparison with unity. Then the quantities of interest may all be expanded for small xy, resulting in a considerable simplification of the problem.

Expressions for x_o and y_o are given for obtaining these quantities by direct calculation, and no computer minimization procedure is required.

By use of this procedure, the problem of treating more than two sets of measurements is no longer formidable. A method for correlating several sets of measurements to obtain the most probable function F(uv) is described in section 4. The method treats equally each set of measurements and generates a set of standard deviations for the shifts by which one may estimate the errors in the measurements and the degree to which the resultant function F is meaningful.

3. RESULTS FOR SMALL DEVIATIONS

If the deviations between two sets of measurements are small, then one may expand the results of the previous section for small x,y and analytically minimize Q(xy). This may be a good preliminary calculation, in any event, even if the deviations are not known to be small or cannot be assumed to be small, since the resultant x_o, y_o may be used as starting values to minimize the full expression for Q given by expression (11).

Expanding expression (11) to second order in x and y, one obtains

$$Q(xy) \cong \sum_{nm} k_{nm} (\pi x)^n (\pi y)^m \tag{17}$$

where

$$k_{00} = (NM)^{-1} \sum_{pq} \left[A(pq) - B(pq) \right]^2 , \tag{18a}$$

$$k_{10} = -2N^{-1} \sum_{p=1}^{N-1} \cot \frac{\pi p}{N} P(p0) , \tag{18b}$$

$$k_{01} = -2M^{-1} \sum_{q=1}^{M-1} \cot \frac{\pi q}{M} P(0q) , \tag{18c}$$

$$\begin{aligned} k_{20} = &-\frac{1}{4} k_{00} + \frac{1}{2} Q_2 + \frac{1}{3}\left(1 + \frac{2}{N^2}\right) P(00) - \frac{3}{2} N^{-1} \sum_p P(p0) \\ &+ \frac{2}{N^2} \sum_{p=1}^{N-1} \left[1 + \cot^2 \frac{\pi p}{N}\right] P(p0) , \end{aligned} \tag{18d}$$

$$\begin{aligned} k_{02} = &-\frac{1}{4} k_{00} + \frac{1}{2} Q_3 + \frac{1}{3}\left(1 + \frac{2}{M^2}\right) P(00) - \frac{3}{2} M^{-1} \sum_q P(0q) \\ &+ \frac{2}{M^2} \sum_{q=1}^{M-1} \left[1 + \cot^2 \frac{\pi q}{M}\right] P(0q) , \end{aligned} \tag{18e}$$

$$k_{11} = -\frac{1}{2}k_{00} + Q_2 + Q_3 - \frac{1}{2}\left(\overline{A} - \overline{B}\right)^2 - N^{-1}\sum_p P(p0) - M^{-1}\sum_q P(0q)$$

$$-2(NM)^{-1}\sum_{p=1}^{N-1}\sum_{q=1}^{M-1} \cot\frac{\pi p}{N}\cot\frac{\pi q}{M}P(pq)\,, \tag{18f}$$

and where Q_2 and Q_3 are given by expressions (12b) and (12c) and P(pq) is defined by expression (14).

Differentiating expression (17) with respect to x and y and separately setting the results equal to zero give the solutions

$$x_0 = \pi^{-1}\left(k_{11}k_{01} - 2k_{10}k_{02}\right) / \left(4k_{20}k_{02} - k_{11}^2\right) \tag{19a}$$

$$y_0 = \pi^{-1}\left(k_{11}k_{10} - 2k_{01}k_{20}\right) / \left(4k_{20}k_{02} - k_{11}^2\right)\,. \tag{19b}$$

Thus, by direct calculation, we have determined the x_0, y_0 that minimize Q. If the resultant values are less than 0.16 ($\pi x < 0.5$), the sine and cosine expansions used to obtain these results approximate the true values to within 1 percent, so the values for x_0, y_0 calculated by this method may be assumed accurate. In the limiting case in which A and B are equal, it can be shown that k_{10} and k_{01} vanish, so x and y are zero as required.

The expansion for R for small x, y is

$$R(p + x/2, q + y/2) \approx A(pq)$$

$$-\left(\frac{\pi x}{2N}\right)\sum_{p'}{}' A(p'q)\cot\frac{\pi}{N}(p'-p) - \left(\frac{\pi y}{2M}\right)\sum_{q'}{}' A(pq')\cot\frac{\pi}{M}(q'-q)$$

$$-A(pq)\left\{\frac{1}{6}\left(\frac{\pi x}{2N}\right)^2(N^2-1) + \frac{1}{6}\left(\frac{\pi y}{2M}\right)^2(M^2-1) + \frac{1}{2}\left[(N-1)\frac{\pi x}{2N} + (M-1)\frac{\pi y}{2M}\right]^2\right\}$$

$$+\frac{\pi x}{2N}\sum_{p'}{}' A(p'q)\left\{\left[N - 1 - \cot^2\frac{\pi}{N}(p'-p)\right]\frac{\pi x}{2N} + (M-1)\frac{\pi y}{2M}\right\}$$

$$+\frac{\pi y}{2M}\sum_{q'}{}' A(pq')\left\{(N-1)\frac{\pi x}{2N} + \left[M - 1 - \cot^2\frac{\pi}{M}(q'-q)\right]\frac{\pi y}{2M}\right\}$$

$$+\frac{\pi x}{2N}\frac{\pi y}{2M}\sum_{p'q'}{}'' A(p'q')\left[\cot\frac{\pi}{N}(p'-p)\cot\frac{\pi}{M}(q'-q) - 1\right] \tag{20}$$

where $\Sigma'_{p'}$ means the p' = p is excluded from the sum; the same holds for q. This expression may not be much simpler to evaluate than the rigorous expression (9), unless x_0 and y_0 are sufficiently small so the quadratic terms may be ignored. In this case,

$$F(pq) \cong \frac{1}{2}\Big[A(pq) + B(pq)\Big]$$

$$-\frac{\pi x_0}{4N} \sum_{p'}{}' \Big[A(p'q) - B(p'q)\Big] \cot\frac{\pi}{N}(p' - p)$$

$$-\frac{\pi y_0}{4M} \sum_{q'}{}' \Big[A(pq') - B(pq')\Big] \cot\frac{\pi}{M}(q' - q) \,. \qquad (21)$$

The rms deviation between the shifted functions from expressions (16) and (17) is given by

$$\sigma = \left[\sum_{nm} k_{nm} \; (\pi x_0)^n \; (\pi y_0)^m\right]^{1/2} \qquad (22)$$

4. CORRELATION OF SEVERAL MEASUREMENT SETS

Suppose there are K sets of measurements $A_i(pq)$ where

$$1 \leqslant i \leqslant K \qquad (23)$$

and each set is shifted by some unknown amount from its presumably true set of values according to the discussion in the previous sections. Let the vector quantity r_{ij} represent the pair of values x_{ij}, y_{ij}, where these are the shifts calculated to minimize the difference between A_i and A_j. That is, the minimum square deviation between A_i and A_j obtains for the extended version of A_i, R_i, evaluated at

$$R_i\,(p + x_{ij}/2,\, q + y_{ij}/2) \qquad (24)$$

and for the extended version of A_j, R_j, evaluated at

$$R_j\,(p - x_{ij}/2,\, q - y_{ij}/2) \,. \qquad (25)$$

Then, clearly

$$r_{ij} = -\, r_{ji} \,. \qquad (26)$$

Now, suppose the centroid of all the measurements is designated by the subscript zero, and r_{i0}, for example, is the shift that must be given to A_i to bring it into line with the centroid. Then

$$\sum_i \; r_{i0} = 0 \qquad (27)$$

is the vector criterion for the centroid. To solve for a given r_{j0}, it is clear that, ideally,

$$r_{i0} = r_{ij} + r_{j0} \,. \qquad (28)$$

Thus expression (27) becomes

$$r_{j0} + \sum_{i \neq j} \; r_{i0}$$

$$= K r_{j0} + \sum_{i \neq j} \; r_{ij} = 0$$

or

$$r_{j0} = K^{-1} \sum_{i \neq j} r_{ji} \tag{29}$$

by use of expression (26). These r_{j0} satisfy expression (27), even though expression (28) may not be strictly correct. It must be stressed that expression (28) is rigorously correct only if the shifted matrices represent the true function that would be measured at the shifted position and no errors existed in the measurements. This assumption, of course, is only approximately correct.

Similarly, although each calculated r_{ij} ideally should equal $r_{in} + r_{nj}$ for any n, taking $r_{ii} = 0$ for all i, errors in the measurements and in the extended functions prevent this equality from being exact. Deviations from equality therefore give some measure of these errors. Accordingly, define $\sigma_{ij}(x)$ by

$$\sigma_{ij}(x) = \left\{ (K-2)^{-1} \sum_{n} \left[x_{ij} - \left(x_{in} + x_{nj} \right) \right]^2 \right\}^{1/2} \tag{30}$$

and define $\sigma_{ij}(y)$ similarly.

Practice will determine how useful the σ_{ij} are, but it is clear that if $\sigma_{ij}(x)$ is not small compared with $|x_{ij}|$, then either some set of measurements that contributed most to the magnitude of $\sigma_{ij}(x)$ should be cast out or the whole procedure of shifting sets of data is questionable for the given data. If the σ_{ij} are reasonably small, however, then

$$F(uv) = K^{-1} \sum_{j} R_j\left(u + x_{j0}, v + y_{j0}\right) \tag{31}$$

is the best extension of the data where the r_{j0} are given by expression (29). For the case of two sets of measurements (K = 2), expression (31) reduces to expression (15) derived in section 2.

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APPENDIX A.–COTANGENT SUMS

From E.T. Whittaker and G.N. Watson, A Course of Modern Analysis, (4th ed., Cambridge University Press, 1958, p. 140, sect. 7.7, example 1),

$$\prod_{i=1}^{K} \cot\frac{\pi}{N}\left(p - a_i\right) = \sum_{m=1}^{K} \prod_{i\neq m} \cot\frac{\pi}{N}\left(a_m - a_i\right) \cot\frac{\pi}{N}\left(p - a_m\right) + \cos\frac{\pi K}{2} \tag{A-1}$$

one may sum over $0 \leqslant p \leqslant N - 1$ to give

$$\sum_p \prod_i \cot\frac{\pi}{N}\left(p - a_i\right) = -N \sum_{m=1}^{K} \prod_{i\neq m} \cot\frac{\pi}{N}\left(a_m - a_i\right) \cot \pi a_m + N \cos\frac{\pi K}{2} \tag{A-2}$$

where one assumes $a_n \neq a_m$ for $n \neq m$ *and* no a_m is an integer. If some of the a_m are integers, the limits of expression (A-2) as each approaches its integer value give

$$\sum_p{}' \prod_i \cot\frac{\pi}{N}\left(p - a_i\right) = -N \sum_{m}^{R} \prod_{i\neq m} \cot\frac{\pi}{N}\left(a_m - a_i\right) \cot \pi a_m$$

$$+ \sum_{m}^{I} \sum_{i\neq m} \prod_{j\neq i\neq m} \cot\frac{\pi}{N}\left(p_m - a_j\right)\left[1 + \cot^2\frac{\pi}{N}\left(p_m - a_i\right)\right] + N\cos\frac{\pi K}{2} \tag{A-3}$$

where Σ'_p extends over $0 \leqslant p \leqslant N - 1$ excluding integers $\mathrm{mod}_N\left(a_m\right)$ for integer a_m, where $\mathrm{mod}_N\left(a_n\right) \neq \mathrm{mod}_N\left(a_m\right)$ for $n \neq m$, where $\sum_m^R$ extends over the noninteger a_m, and where $\sum_m^I$ extends over the integer a_m.

From expressions (A-2) and (A-3), then, for K = 1,

$$\sum_p \cot\frac{\pi}{N}\left(p - a\right) = -N \cot \pi a\ , \tag{A-4a}$$

$$\sum_{p\neq p_1} \cot\frac{\pi}{N}\left(p - p_1\right) = 0\ , \tag{A-4b}$$

and for K = 2, where $a_1 \neq a_2 \neq$ integer, $p_1 \neq p_2$,

$$\sum_p \cot\frac{\pi}{N}\left(p - a_1\right) \cot\frac{\pi}{N}\left(p - a_2\right) = -N - N\cot\frac{\pi}{N}\left(a_1 - a_2\right)\left[\cot \pi a_1 - \cot \pi a_2\right]\ , \tag{A-5a}$$

$$\sum_{p\neq p_1} \cot\frac{\pi}{N}\left(p - p_1\right) \cot\frac{\pi}{N}\left(p - a_2\right) = -N + N\cot\frac{\pi}{N}\left(p_1 - a_2\right) \cot \pi a_2 + 1 + \cot^2\frac{\pi}{N}\left(p_1 - a_2\right)\ . \tag{A-5b}$$

$$\sum_{p\neq p_1\neq p_2} \cot\frac{\pi}{N}\left(p - p_1\right) \cot\frac{\pi}{N}\left(p - p_2\right) = -N + 2\left[1 + \cot^2\frac{\pi}{N}\left(p_1 - p_2\right)\right]\ . \tag{A-5c}$$

Differentiating expression (A-4a) with respect to "a," one may generate

$$\sum_p \left[\cot \frac{\pi}{N} (p-a)\right]^K = f_K(a) \tag{A-6}$$

where f satisfies

$$f_{K+1} = -f_{K-1} + \frac{N}{K\pi} f'_K \tag{A-7}$$

with

$$f_0 = N;\ f_1 = -N \cot \pi a \tag{A-8}$$

and $f' = df/da$. Thus, solving expression (A-7) for K = 1,

$$\sum_p \left[\cot \frac{\pi}{N}(p-a)\right]^2 = -N + N^2\left[1 + \cot^2 \pi a\right] \tag{A-9a}$$

for example. Limiting expression (A-9a) as $a \to p_1$ gives

$$\sum_{p \neq p_1} \left[\cot \frac{\pi}{N}(p-p_1)\right]^2 = \frac{1}{3}(N-1)(N-2)\,. \tag{A-9b}$$

Thus, the generalization of expression (A-5c) is

$$\sum_{\substack{p \neq p_1 \\ p \neq p_2}} \cot \frac{\pi}{N}(p-p_1)\cot\frac{\pi}{N}(p-p_2) = -N + 2\left(1-\delta_{p_1p_2}\right)\left[1 + \cot^2 \frac{\pi}{N}(p_1-p_2)\right]$$
$$+\frac{1}{3}\left(N^2+2\right)\delta_{p_1p_2}\,. \tag{A-10}$$

Expression (A-6) may be generalized by differentiating expression (A-2) q_n times with respect to a_n for each n to develop a sum of the form

$$\sum_p \prod_m \left[\cot\frac{\pi}{N}(p-a_m)\right]^{q_m} = f_{q_1 \ldots q_k}\left(a_1 \ldots a_k\right). \tag{A-11}$$

By use of

$$\frac{d}{da}\cot\frac{\pi}{N}(p-a) = \frac{\pi}{N}\left[1 + \cot^2\frac{\pi}{N}(p-a)\right], \tag{A-12}$$

the following recursion expression for the generalized f is obtained,

$$f_{q_1 \ldots q_n + 1 \ldots q_k} = -f_{q_1 \ldots q_n - 1 \ldots q_k} + \frac{N}{\pi q_n}\frac{d}{da_n} f_{q_1 \ldots q_n \ldots q_k}\,, \tag{A-13}$$

where the boundary conditions for f if each q = 0 or 1 are obtained from expression (A-2). A limiting procedure is taken to evaluate expression (A-11) if some of the a_n are integers. The result for a given a_n = integer, say $a_n = p_n$, is obtained by

$$\sum_{p \neq p_n} \prod_m \left[\cot \frac{\pi}{N}\left(p - a_m\right)\right]^{q_m}$$

$$= \lim_{a_n \to p_n} \left\{ f_{q_1 \ldots q_k}\left(a_1 \ldots a_n \ldots a_k\right) - \prod_m \left[\cot \frac{\pi}{N}\left(p_n - a_m\right)\right]^{q_m} \right\} . \tag{A-14}$$

Similarly, additional limits may be taken if several of the a_n are integers. Note that if $p_n > N - 1$, then let $a_n \to \mathrm{mod}_N\left(p_n\right)$, since $f\left(a_n + N\right) = f\left(a_n\right)$ independently for each a_n.

For K = 2,

$$\sum_p \left[\cot \frac{\pi}{N}\left(p - a_1\right)\right]^{q_1} \left[\cot \frac{\pi}{N}\left(p - a_2\right)\right]^{q_2} = f_{q_1 q_2}\left(a_1, a_2\right) \tag{A-15}$$

where

$$f_{q_1 q_2}\left(a_1, a_2\right) = f_{q_2 q_1}\left(a_2, a_1\right) \tag{A-16}$$

and

$$f_{00} = N; \; f_{10} = -N \cot \pi a_1 \; , \tag{A-17a}$$

$$f_{11} = -N - N \cot \frac{\pi}{N}\left(a_1 - a_2\right)\left[\cot \pi a_1 - \cot \pi a_2\right] \; , \tag{A-17b}$$

$$f_{20} = -N + N^2 \left[1 + \cot^2 \pi a_1\right] \; , \tag{A-17c}$$

$$f_{21} = -N \cot \pi a_2 \cot^2 \frac{\pi}{N}\left(a_1 - a_2\right)$$

$$+ N \left[\frac{\cot \pi a_1}{\sin^2 \frac{\pi}{N}\left(a_1 - a_2\right)} + \frac{N \cot \frac{\pi}{N}\left(a_1 - a_2\right)}{\sin^2 \pi a_1}\right] \; , \tag{A-17d}$$

$$f_{22} = N + 2N \cot \frac{\pi}{N}\left(a_1 - a_2\right)\left[\frac{\cot \pi a_1 - \cot \pi a_2}{\sin^2 \frac{\pi}{N}\left(a_1 - a_2\right)}\right]$$

$$+ N^2 \cot^2 \frac{\pi}{N}\left(a_1 - a_2\right)\left[2 + \cot^2 \pi a_1 + \cot^2 \pi a_2\right] \tag{A-17e}$$

APPENDIX B. – THE C_N (p,z) FUNCTION

In the form

$$C_N(p,z) = N^{-1} \sin \pi z \left[\cot \frac{\pi}{N} (p + z) + i\right] , \tag{B-1}$$

the following properties of these functions may be derived from a limiting procedure of the cotangent sums of appendix A:

$$C_N(p,0) = \delta_{p0} , \tag{B-2}$$

$$C_N(p', p + z) = (-)^p \, C_N (p' + p, z) , \tag{B-3}$$

$$C_N^*(p,z) = C_N(-p, -z) \tag{B-4}$$

$$= C_N(p,z) - 2N^{-1} \, i \sin \pi z , \tag{B-5}$$

$$\sum_p C_N(p, z) = \sum_p C_N(-p, z) = \sum_p C_N(p + p', z) = e^{i\pi z}, \tag{B-6}$$

$$\sum_p C_N\left(p - p', z\right) C_N^*\left(p - p'', z'\right) = C_N\left(p'' - p', z - z'\right) , \tag{B-7}$$

$$\sum_p C_N\left(p - p', z\right) C_N\left(p - p'', z'\right) = C_N\left(p'' - p', z - z'\right) + 2iN^{-1} \, e^{i\pi z} \sin \pi z' \tag{B-8}$$

$$= C_N\left(p' - p'', z' - z\right) + 2iN^{-1} \, e^{i\pi z'} \sin \pi z . \tag{B-9}$$

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